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A DIRECT APPROACH TO SECOND-ORDER MATRIX NON-CLASSICAL VIBRATING EQUATIONS

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Abstract

A direct framework is developed for second-order matrix equations without transforming it into a first-order companion equation. It is done in terms of its matrix impulse response that is directly related to the transfer matrix. We formulate an extension of the Cayley-Hamilton identity, derive the controllability and observability matrices and discuss Krylov's method in terms of such matrix response. This formulation will allow to further discuss Arnoldi and Lanczos methods as well as time-integration by FFT

1 Introduction

We develop a direct framework for the second-order matrix equation

$$Mq''(t) + Cq'(t) + Kq(t) = f(t), \tag{1}$$

without transforming it into a first-order equation. This is acomplished in terms of its matrix impulse response and it is illustrated through an extended Cayley-Hamilton identity, a direct derivation of the controllability and observability matrices, and a direct formulation of a Krylov subspace method for second-order equations. The results can be conveniently translated for the discrete analogue

$$Mq_{k+2} + Cq_{k+1} + Kq_k = f_k. \tag{2}$$

Here M, C and K are arbitrary $n \times n$ matrices with numerical elements, M non-singular, and $q(t), f(t)$ real functions with values in the n-dimensional euclidean space.

The standard approach for studying such systems has been a transformation into an equivalent first-order system. However, this implies putting aside the "mechanical"or physical coordinates which are of increasing interest in control problems and natural for measurements.

The treatment given here is based on the Laplace transform. This allows to relate directly the impulse matrix response ("transition matrix") to the transfer matrix. An explicit formulation of the impulse matrix response is given in terms of a scalar characteristic differential equation and its associated second-order discrete analogue. This allows to establish several properties of second-order systems. For instance, it allows to extend properties such as the Cayley-Hamilton identity, semigroup type relationships, and a characterization of eigenvectors. This approach can be easily generalized to higher-order continuous and discrete equations.

2 The second-order state framework

In this section, we shall treat second-order equations in their own framework, that is, without relying on the reduction to a first-order system through a companion matrix formulation. By applying the Laplace transform to the non-homogeneous equation

$$Mq''(t) + Cq'(t) + Kq(t) = f(t), \tag{3}$$

we obtain the operational equation

$$\Delta(s)Q(s) = (sM + C)q(0) + Mq'(0) + F(s),$$

where

$$\Delta(s) = s^2M + Cs + K.$$

Thus

$$Q(s) = H(s)[(sM + C)]q(0) + H(s)Mq'(0) + H(s)F(s),$$

where

$$[s^2M + sC + K]H(s) = I.$$

We have that $H(s)$ is the Laplace transform of the matrix solution satisfying

$$Mh''(t) + Ch'(t) + Kh(t) = 0, \tag{4}$$

$$Mh'(0^+) = I\ , \quad h(0^+) = 0,$$

where I denotes the matrix identity. By taking Laplace inverse transform we obtain the variation of constants formula

$$q(t) = h_0(t)q(0) + h(t)Mq'(0) + \int_0^t h(t-s)f(s)ds, \tag{5}$$

where

$$h_0(t) = h'(t)M + h(t)C. \tag{6}$$

We shall refer to $h(t)$ as the impulse matrix response, also refered to as the dynamical solution or the state-transition matrix of the system. Since $H(s)$ commutes with the matrix polynomial $\Delta(s)$, we have that $h(t)$ is also a left solution of the homogeneous equation, that is

$$h''(t)M + h'(t)C + h(t)K = 0, \quad h'(0^+)M = I, \quad h(0^+) = 0. \tag{7}$$

Although the forcing term is assumed to be Laplace transformable, by direct substitution on the given equation, the validity of the variations of constants formula can be extended to a larger class of forcing terms.

From a physical point of view, we have that a matrix impulse $f(t) = \delta(t)I$ applied to a second-order system initially at rest, imparts a change in momentum. Thus the matrix response can be thought of as the response of a system which has been in the zero state for $t \leq 0$, when a unit impulse is applied at $t = 0$, that is

$$Mh''(t) + Ch'(t) + Kh(t) = \delta(t)I, \quad Mh'(0) = 0, \quad h(0) = 0. \tag{8}$$

The elements $h_{kj}(t)$ of the matrix impulse response are to be interpreted as being the response of the k-th component of the system due to a unit initial momentum (or unit impulse force) at the j-th component.

It can be easily verified that $h_0(t)$ satisfies the same matrix equation as $h(t)$, but with initial values $Mh_0'(0) = 0, \ \ h_0(0) = I$. However, $h_0(t)$ is not a left solution unless the matrix coefficients commute themselves [4].

For the discrete equation

$$Mq_{k+2} + Cq_{k+1} + Kq_k = f_k, \tag{9}$$

we have that

$$q_k = [h_{k+1}M + h_kC]q_0 + h_kMq_1 + \sum_{j=0}^{k-1} h_{k-1-j}f_j, \tag{10}$$

where $h_k = h^{(k)}(0)$ is the discrete matrix impulse response, that is

$$Mh_{k+2} + Ch_{k+1} + Kh_k = h_{k+2}M + h_{k+1}C + h_kK = 0, \tag{11}$$

with the initial values $Mh_1 = I$, $h_0 = 0$. This follows by applying the z-transform to the given discrete equation or by a simple identification of q_k as the k-th derivative of the solution of the continuous time equation for a given analytic forcing term.

2.1 A rational matrix characterization of the transfer function

In order to derive a formula for the impulse matrix response, we need the following characterization of the transfer function

$$H(s) = (s^2M + Cs + K)^{-1} = \sum_{j=1}^{2n}\sum_{i=0}^{j-1} b_i \frac{s^{j-i-1}}{P(s)} h_{2n-j}, \tag{12}$$

where

$$P(s) = det[s^2M + sC + K] = \sum_{k=0}^{2n} b_k s^{2n-k} \tag{13}$$

is the associated characteristic polynomial and h_k the discrete matrix impulse response.

The proof follows via a convenient use of the Cramer identity

$$B(s)\Delta(s) = \Delta(s)B(s) = det(\Delta(s))I = P(s)I,$$

where the adjugate matrix $B(s)$ (transposed matrix formed from the cofactors of Δ) is a polynomial of degree less than $2n-1$. By differentiating j times the Cramer identity and noticing that $\Delta(0) = K$, $\Delta'(0) = C$ and $\Delta''(0) = 2M$, it turns out that for $j > 2$, the matrix coefficients $B_j = B^{(j)}(0)$ satisfy the difference equation

$$\begin{aligned} &MA_{j+2} + CA_{j+1} + KA_j = b_jI, \quad j = 0, \ldots 2n-2, \\ &A_0 = A_1 = O, \end{aligned}$$

where $A_j = \frac{B_{2n-j}}{(2n-j)!}$.

The solution of this equation is given by the convolution

$$A_j = \sum_{i=0}^{j-1} b_i h_{j-i-1},$$

where, as before $Mh_{k+2} + Ch_{k+1} + Kh_k = 0; \quad Mh_1 = I, \ h_0 = 0.$

Since $H(s) = \frac{B(s)}{P(s)}$, the result follows by substituting the values B_{2n-j} or A_j on the Taylor expansion for $B(s)$.

We can substitute $s = i\omega$ to obtain the *frequency response*

$$H(i\omega) = \sum_{j=1}^{2n} \sum_{k=0}^{j-1} b_k \frac{(i\omega)^{j-1-k}}{P(i\omega)} h_{2n-j}. \tag{14}$$

2.2 The impulse matrix response

The characterization of the transfer function of a second-order matrix equation allows to recover the impulse matrix response by taking the Laplace inverse transform. For doing so, we shall introduce the following characteristic differential equation

$$b_0 d^{(2n)}(t) + b_1 d^{(2n-1)}(t) + \cdots + b_{2n-1} d'(t) + b_{2n} d(t) = 0, \tag{15}$$

with the initial data

$$b_0 d^{(2n-1)}(0) = 1, \quad d^{(2n-2)}(0) = \cdots = d'(0) = d(0) = 0, \tag{16}$$

whose solution can be written as the Bromwich integral [2]

$$d(t) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{e^{st}}{P(s)} ds.$$

The corresponding scalar transfer function

$$\phi(s) = \frac{1}{P(s)},$$

is such that $\frac{s^k}{P(s)}$ corresponds to the Laplace transform of the k-th derivative of $d(t)$. This allows to conclude that the impulse matrix response $h(t)$ is given by

$$h(t) = \sum_{j=1}^{2n} \sum_{i=0}^{j-1} b_i d^{(j-i-1)}(t) h_{2n-j}, \tag{17}$$

where

$$Mh_{k+2} + Ch_{k+1} + Kh_k = h_{k+2}M + h_{k+1}C + h_k K = 0, \tag{18}$$

with the initial values $Mh_{2n-1} = I, \quad h(0) = 0.$

By noticing that

$$sH(s) = s(s^2 I + Cs + K)^{-1} = \frac{sB(s)}{P(s)} = \sum_{j=1}^{2n} \sum_{i=0}^{j-1} b_i \frac{s^{j-i-1}}{P(s)} h_{2n-j+1}, \tag{19}$$

we arrive to the following shifting property for the derivative of the impulse matrix response

$$h'(t) = \sum_{j=0}^{2n} \sum_{i=0}^{j-1} b_i d^{(j-i-1)}(t) h_{2n-j+1}. \tag{20}$$

2.3 An extended Cayley-Hamilton identity

We now proceed to establish an extension of the Cayley-Hamilton identity for second-order pencils $s^2 M + sC + K$. With the same notation as before, we have that

$$\sum_{i=0}^{2n} b_i h_{2n-i} = 0. \tag{21}$$

As we shall see below, this property is related to a shifting property of the impulse response $h(t)$. The proof follows by writting the derivative of $h(t)$

$$h'(t) = \sum_{j=1}^{2n} \sum_{i=0}^{j-1} b_i d^{(j-i)}(t) h_{2n-j},$$

as

$$h'(t) = \sum_{j=2}^{2n+1} \sum_{i=0}^{j-2} b_i d^{(j-i-1)}(t) h_{2n-j+1} = \sum_{j=1}^{2n} (\sum_{i=0}^{j-1} b_i d^{(j-i-1)}(t) - b_{j-1} d(t)) h_{2n-j+1},$$

or simply

$$h'(t) = \sum_{j=1}^{2n} \sum_{i=0}^{j-1} b_i d^{(j-i-1)}(t) h_{2n-j+1} - \sum_{j=0}^{2n-1} b_j d(t)) h_{2n-j}.$$

By the shifting property, we have that the first term on the right hand side is precisely $h'(t)$, from which it follows the proposed extension.

Remark We should point out that the extended Cayley-Hamilton identity is also valid for h_{2n-i+k}, $k = 0, 1, 2, \ldots$, instead of h_{2n-i}, that is

$$\sum_{i=0}^{2n} b_i h_{2n-i+k} = 0. \tag{22}$$

The reason being that the *shifting property* is valid for the derivatives of arbitrary order of $h(t)$. More precisely,

$$h^{(k)}(t) = \sum_{j=o}^{2n} \sum_{i=0}^{j-1} b_i d^{(j-i-1)}(t) h_{2n-j+k}. \tag{23}$$

This will be shown after we establish further properties for the impulse matrix response by using the companion matrix approach.

2.4 The case of simple roots

By introducing the following polynomials

$$q_j(s) = \sum_{i=0}^{j-1} b_i s^{j-1-i}, \quad j = 1, 2, \cdots, 2n, \tag{24}$$

we can write

$$H(s) = \sum_{j=1}^{2n} \sum_{i=0}^{j-1} \frac{q_j(s)}{P(s)} h_{2n-j}.$$

Let s_k be a root of the characteristical polynomial with multiplicity m_k. If we let $r_{ij}(s_k)$ to be the residue of $\frac{q_j(s)}{P(s)}$ evaluated at the root s_k, we shall have that

$$H(s) = \sum_{k=1}^{r} \sum_{i=1}^{m_k} \frac{E_{ki}}{(s - s_k)^{m_k - i - 1}},$$

where

$$E_{ki} = \sum_{j=1}^{2n} r_{ji}(s_k) h_{2n-j}.$$

When the characteristic polynomial has simple roots, $m_k = 1$, it turns out that

$$h(t) = \sum_{k=1}^{2n} E_k e^{s_k t}, \qquad H(s) = \sum_{k=1}^{2n} \frac{E_k}{s - s_k}, \tag{25}$$

where E_k is now given by

$$E_k = \frac{1}{p'(s_k)} \sum_{j=1}^{2n} q_j(s_k) h_{2n-j}. \tag{26}$$

3 Relationships with the companion matrix

The treatment of second-order mechanical systems is usually done by introducing the Hamilton state formulation

$$x_1 = q, \quad x_2 = q'; \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

This approach allows to translate several properties to our second-order framework [4].

Let us consider the first-order system

$$x' = Ax + F(t), \tag{27}$$

where

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}, \tag{28}$$

is the companion matrix of order $2n \times 2n$, x the state vector $2n \times 1$ and $F(t) = col[0 \;\; f(t)]$ a column forcing vector $2n \times 1$.

Exponential solutions $x = e^{t\lambda}v$ give rise to eigenvectors of of the form

$$y = \begin{bmatrix} v \\ \lambda v \end{bmatrix}, \tag{29}$$

where v is an eigenvector of the second-order system and λ its eigenvalue, that is, $[\lambda^2 M + \lambda C + K]v = 0$.

With such formulation it is easy to show that

$$e^{tA} = \begin{bmatrix} h_0(t) & h(t)M \\ h_0'(t) & h'(t)M \end{bmatrix}. \tag{30}$$

By differentating the exponential matrix at the origin, we obtain the following block representation for the powers of the companion matrix

$$A^k = \begin{bmatrix} h_{o,k} & h_k M \\ h_{0,k+1} & h_{k+1}M \end{bmatrix}, \tag{31}$$

where $h_{0,k} = h_0^{(k)}(0)$ satisfies the matrix difference equation $Mq_{k+2} + Cq_{k+1} + Kq_k = 0$ with initial values $q_0 = I$ and $q_1 = 0$.

The following properties are an extension of the ones for *sine* and *cosine* matrix functions that appear in conservative systems ($C = 0$)

$$\begin{aligned} h(t+s) &= h_0(t)h(s) + h(t)Mh'(s), \\ h'(t+s) &= h_0'(t)h(s) + h'(t)Mh'(s). \end{aligned} \tag{32}$$

They follow from the exponential semigroup property and the block characterization in terms of the impulse matrix response given above. We shall refer to these properties as the *extended semigroup property* for second-order systems.

By differentiating the exponential matrix and using the formula for the powers of A, we arrive to the general shifting property of the impulse matrix response $h(t)$

$$h^{(k)}(t) = \sum_{j=1}^{(2n)} \sum_{i=0}^{j-1} b_i d^{(j-1-i)}(t) h_{2n-j+k}.$$

The block characterization of the powers of the companion matrix, in terms of this discrete matrix impulse response, and the Cayley-Hamilton identity with such matrix, allows us to obtain the extended Cayley-Hamilton identity for a second-order matrix equation

$$\sum_{i=0}^{2n} b_i h_{2n-j+p} = 0, \quad p = 0, 1, 2, \cdots .$$

We should observe that the validity of such identity is analogous to the situation of multiplying the Cayley-Hamilton identity of a matrix A by any power of it.

4 Controllability and observability matrices

Here we shall derive the controllability and observability matrices for second-order systems by employing the impulse matrix response. This approach is a direct extension of the arguments employed with first-order systems. There is no need to use the standard companion matrix approach.

Let us consider the second-order control system

$$Mq'' + Cq' + Kq = Bu \tag{33}$$

where M, C, K, are arbitrary $n \times n$ matrices, M non-singular, and B a control matrix of order $n \times m$. If the system is controllable, then there is a control u and a time t such that we can drive q and q' to the origin. By using the variation of constants formula, this means that

$$0 = h_0(t)q(0) + h(t)Mq'(0) + \int_0^t h(t-s)Bu(s)ds,$$

$$0 = h_0'(t)q(0) + h'(t)Mq'(0) + \int_0^t h'(t-s)Bu(s)ds,$$

or simply

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} h_0(t) & h(t)M \\ h_0'(t) & h'(t)M \end{bmatrix} \begin{bmatrix} q(0) \\ q'(0) \end{bmatrix} + \int_0^t \begin{bmatrix} h(t-s) & Bu(s) \\ h'(t-s) & Bu(s) \end{bmatrix} ds. \tag{34}$$

We now use the extended semigroup property on the integral so that we can factor out a common matrix block which is non-singular, since it is the matrix exponential. Thus

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} q(0) \\ q'(0) \end{bmatrix} + \int_0^t \begin{bmatrix} h(-s)Bu(s) \\ h'(-s)Bu(s) \end{bmatrix} ds. \tag{35}$$

By substituting

$$h(t) = \sum_{j=1}^{2n} \beta_j(t) h_{2n-j}, \quad h(t) = \textstyle\sum_{j=1}^{2n} \beta_j(t) h_{2n-j+1},$$

with

$$\beta_j(t) = \sum_{i=0}^{j-1} b_i d^{(j-1-i)}(t),$$

we obtain the algebraic linear system

$$E\alpha = w,$$

where

$$w = -\begin{bmatrix} q(0) \\ q'(0) \end{bmatrix} \quad , \quad \alpha = col[\alpha_1 \ \ \alpha_2 \ \ \cdots \ \ \alpha_{2n}], \quad \alpha_i(t) = \int_0^t \beta_i(-s)u(s)ds,$$

and E turns out to be the **controllability matrix**

$$E = \begin{bmatrix} h_0 B & h_1 B & \cdots & h_{2n-1} B \\ h_1 B & h_2 B & \cdots & h_{2n} B \end{bmatrix} \tag{36}$$

for a second-order matrix system. If the system is controllable, then the given equation will have a solution for arbitary w. This implies that rank $E = 2n$.

The converse of the above rank condition is also true, that is, if the matrix E has rank $2n$ then the second-order equation is controllable. This amounts to show that there is a control u and a time t_1 such that the **controllability Grammian**

$$W_C = \int_0^{t_1} \begin{bmatrix} h(-s)B \\ h'(-s)B \end{bmatrix} \begin{bmatrix} h(-s)B \\ h'(-s)B \end{bmatrix}^t ds \tag{37}$$

is non-singular. The proof is entirely similar to the one found in the literature for first-order equations and we shall not repeat it here. In such a case, the control

$$u(t) = Q(t)W_C^{-1}\left\{ \begin{bmatrix} h_0(-t_1) & h(-t_1)M \\ h_0'(-t_1) & h'(-t_1)M \end{bmatrix} \begin{bmatrix} q(t_1) \\ q'(t_1) \end{bmatrix} - \begin{bmatrix} q(0) \\ M^{-1}q'(0) \end{bmatrix} \right\},$$

where

$$Q(t) = \begin{bmatrix} h(-t)B \\ h'(-t)B \end{bmatrix}^t,$$

will be such that

$$\int_0^{t_1} Q(s)u(s)ds = \begin{bmatrix} h_0(-t_1) & h(-t_1)M \\ h_0'(-t_1) & h'(-t_1)M \end{bmatrix} \begin{bmatrix} q(t_1) \\ q'(t_1) \end{bmatrix} - \begin{bmatrix} q_0 \\ M^{-1}q_0' \end{bmatrix}.$$

By using the extended semigroup property and the fact that $e^{-At}e^{At} = I$, we obtain

$$\begin{bmatrix} q(t_1) \\ \\ q'(t_1) \end{bmatrix} = \begin{bmatrix} h_0(t_1) & h(t_1) \\ \\ h_0'(t_1) & h(t_1) \end{bmatrix} \begin{bmatrix} q_0 \\ \\ Mq_0' \end{bmatrix} + \int_0^{t_1} \begin{bmatrix} h(t_1 - s)Bu(s) \\ \\ h_0'(t_1 - s)Bu(s) \end{bmatrix} ds.$$

That is, the initial state can be transfered to another state in finite time through the above control u.

It should be observed that the controllability matrix and the Grammian could have been derived by a convenient use of our formula for the powers of a companion matrix, or by the block characterization of the exponential corresponding to the equivalent first-order equation with the companion matrix. However, we would have missed a key point, that is, the shifting property, which is essential when using the exponential matrix with a first-order equation.

Remark

The controllability matrix

$$\begin{bmatrix} h_0B & h_1B & \cdots & h_{2n-2}B & h_{2n-1}B \\ h_1B & h_2B & \cdots & h_{2n-1}B & h_{2n}B \end{bmatrix}$$

was derived in [8] by using a Jordan type of spectral factorization for matrix polynomials. Here that matrix has been derived by a non-spectral direct method which resembles Kalman's work.

4.1 The observability matrix

We now consider the control problem

$$\begin{aligned} Mq'' + Cq' + Kq &= Bu, \\ y &= Fq + Pq', \end{aligned} \tag{38}$$

where M, C, K, are $n \times n$ matrices, M is non-singular, B is $n \times m$ and F, P are $s \times n$. The above system is observable when from the knowlegde of the output $y(t)$ and $u(t)$, M, C, K, F, P, it is possible to determine the initial state . From the variation of constants formula, we have that the control forcing term contributes a known convolution response term to the observed value. Thus, we may as well consider $B = 0$. We can write

$$y = F[h_0(t)q(0) + h(t)Mq'(0)] + P[h_0'(t)q(0) + h'(t)Mq'(0)],$$

or simply

$$y = [F(h'M + hC) + P(h''M + h'C)]q(0) + [Fh + Ph']Mq'(0).$$

By substituting the formulas for h and h', it turns out

$$y = \sum_{j=0}^{2n-1} \beta_j(t)\{[F(h_{j+1}M + h_jC) + P(h_{j+2}M + h_{j+1}C)]q(0) + [Fh_j + Ph_{j+1}]Mq'(0)\}.$$

In matrix terms, we have

$$y = \beta\; R\; Q_0,$$

where

$$\beta = [\beta_0 \;\; \beta_1 \;\; \cdots \;\; \beta_{2n-1}],$$

$$R = \begin{bmatrix} F(h_1M + h_0C) + P(h_2M + h_1C) & (Fh_0 + Ph_1)M \\ F(h_2M + h_1C) + P(h_3M + h_2C) & (Fh_1 + Ph_2)M \\ F(h_3M + h_2C) + P(h_4M + h_3C) & (Fh_2 + Ph_3)M \\ \vdots & \vdots \\ F(h_{2n}M + h_{2n-1}C) + P(h_{2n+1}M + h_{2n}C) & (Fh_{2n-1} + Ph_{2n})M \end{bmatrix},$$

$$Q_0 = \begin{bmatrix} q_0 \\ \cdot \\ q_0' \end{bmatrix}.$$

This system will have a solution for an arbitrary initial state whenever the rank of the observability matrix R is equal to $2n$. Since

$$h_{k+2}M +_{hk+1} C = -h_k K,$$

the observability matrix R can be further simplified. Finally, the observability Grammian can be also introduced in the usual way.

5 Krylov subspace method

Here we shall discuss Krylov's methods for computing the coeficients of a characteristic polynomial and the eigenvectors associated with a second-order matrix pencil $s^2M + sC + K$.

Let v be a non-zero vector and set $v_k = h_k v$. From the extended Cayley-Hamilton identity we have

$$\sum_{k=0}^{2n} b_k h_{2n-k+p} v_{\cancel{k}} = \sum_{k=0}^{2n} b_k v_{2n-k+p} = 0, \quad p = 0, 1.$$

We write in matrix form

$$\begin{bmatrix} v_{2n-1} & v_{2n-2} & \cdots & v_1 & v_0 \\ v_{2n} & v_{2n-1} & \cdots & v_2 & v_1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{2n-1} \\ b_{2n} \end{bmatrix} = -\tfrac{1}{b_0}\begin{bmatrix} v_{2n} \\ v_{2n+1} \end{bmatrix}.$$

The matrix on the left, with the above definition of v_k in terms of h_k, can be written

$$\begin{bmatrix} h_{2n-1}v & h_{2n-2}v & \cdots & h_1v & h_0v \\ h_{2n}v & h_{2n-1}v & \cdots & h_2v & h_1v \end{bmatrix}.$$

This is just a block permutation of the controllability matrix for the second-order equation with a scalar control $u(t)$

$$q'' + Cq' + Kq = vu(t).$$

We thus conclude that *if the above system is controllable for certain nonzero vector v, then the coefficients of the characteristic polynomial can be obtained by Krylov's method.*

The eigenvector equation

$$(s^2M + sC + K)v = 0, \quad v \neq 0 \tag{39}$$

arises when seeking non-zero exponential solutions $q = e^{\lambda t}v$ for the second-order equation $Mq''+Cq'+Kq = 0$. By writting the exponential solution in terms of the matrix impulse response h_k, it turns out that v satisfies the equivalent relationship

$$(h_{k+1}M + h_kC)v + \lambda h_kMv = \lambda^k v, \tag{40}$$

for arbitrary non-negative integer k.

Let us assume that all eigenvalues λ_k, with corresponding eigenvectors v_k, are distinct. Then for any given vectors y_0 and y_1 we can find constants c_k such that

$$y_0 = c_1v_1 + c_2v_2 + \cdots + c_2nv_{2n},$$

$$y_1 = c_1\lambda_1v_1 + c_2\lambda_2v_2 + \cdots + c_n\lambda_2nv_{2n}.$$

The reason being that the vectors

$$w_k = \begin{bmatrix} v_k \\ \lambda_kv_k \end{bmatrix}$$

constitute a basis for the $2n$-dimensional euclidean space.

Let us define the vectors

$$y_k = (h_{k+1}M + h_kC)y_0 + h_{k+1}My_1. \tag{41}$$

By substituting y_0 and y_1, it follows that

$$y_k = c_1\lambda_1^kv_1 + c_2\lambda_2^kv_2 + \cdots + c_n\lambda_{2n}^kv_{2n}, \quad k = 0 : 2n-1.$$

We now consider the linear combination

$$q_{1,i}y_{2n-1} + q_{2,i}y_{2n-2} + \cdots + q_{2n,i}y_0 = \sum_{k=1}^{2n} c_k\phi_i(\lambda_k)v_k,$$

where

$$\phi_i(\lambda) = \sum_{r=1}^{2n} q_{r,i}\lambda^{2n-r}.$$

By choosing

$$\phi_i(\lambda) = \frac{P(\lambda)}{\lambda - \lambda_i},$$

an eigenvector corresponding to the eigenvalue λ_i can be obtained as the linear combination

$$v_i = \sum_{j=1}^{2n} q_j(\lambda_i) y_{2n-j}, \tag{42}$$

where $q_{j,i} = q_j(\lambda_i)$. Here the polynomials

$$q_j(\lambda) = \sum_{k=0}^{j-1} b_k \lambda^{j-1-k},$$

can be obtained by the recurrence relation $q_0(\lambda) = b_0, \quad q_j(\lambda) = \lambda q_{j-1}(\lambda) + b_j$.

Remark

By choosing $y_0 = 0$ and y_1 to be the first vector of the usual basis for the euclidean space, we obtain that v_i is just the eigenvector obtained by Danilevskii method [6].

6 Difference and higher-order matrix equations

The results obtained for second-order matrix equations can be easily generalized for higher-order or discrete equations. We shall simply enunciate them since the arguments will the same as those employed for second-order equations.

Let us consider the m-th order equation

$$A_0 q^{(m)}(t) + A_1 q^{(m-1}(t) + \cdots + A_{m-1} q'(t) + A_m q(t) = f(t), \tag{43}$$

where the $A_k's$ are $n \times n$ scalar matrices, with A_0 non-singular.

The transfer matrix is given by

$$H(s) = [\sum_{k=0}^{mn} A_k s^{mn-k}]^{-1} = \sum_{j=1}^{mn} \sum_{i=0}^{j-1} b_i \frac{s^{j-i-1}}{P(s)} h_{mn-j}, \tag{44}$$

where

$$P(s) = det[\sum_{i=0}^{m} A_i s^{m-i}] = \sum_{i=0}^{mn} b_i s^{mn-i}, \tag{45}$$

and h_k is the solution of the characteristic difference equation

$$\sum_{j=0}^{m} A_j h_{(m+k-j)} = 0, \quad M h_{m-1} = I, \quad h_j = 0, \quad j = 0 : m-2. \tag{46}$$

As before, $h(t)$ is the inverse Laplace transform of the transfer matrix and satisfies

$$A_0 h^{(m)}(t) + A_1 h^{(m-1}(t) + \cdots + A_{m-1} h'(t) + A_m h(t) = 0, \tag{47}$$

with the initial values $A_0 h^{(m-1)} = I, \quad h^{(j)}(0) = 0 \text{ for} j = 0 : m-2$.

We have that

$$h^{(p)}(t) = \sum_{j=1}^{mn} \sum_{i=0}^{j-1} b_j d^{(j-i-1)}(t) h_{mn-j+p}, \quad p = 0, 1, 2, \cdots, \tag{48}$$

where $d(t)$ is the solution of the scalar characteristic differential equation

$$\sum_{k=0}^{mn} b_j d_{(j)}(t) = 0, \quad b_0 d_{mn-1} = 1, \quad d^{(j)}(0) = 0, \ j = 0 : mn-2. \tag{49}$$

The extended Cayley-Hamilton identity reads

$$\sum_{i=0}^{mn} b_i h_{mn-i+p} = 0, \quad p = 0, 1, 2, \cdots \tag{50}$$

The case of difference equations can be handled directly with the z-transform or by using the power series relationship $q_k = q^{(k)}(0)$. Let us consider the discrete equation

$$A_0 q_{m+k} + A_1 q_{m+k-1} + \cdots + A_{m-1} q_{k+1} + A_m q_k = f_k. \tag{51}$$

We have that the *discrete* matrix impulse response $h_k = h^{(k)}(0)$ satisfies

$$\sum_{j=0}^{m} A_j h_{k+m-j} = 0, \quad A_0 h_{m-1} = I, \quad h_j = 0, \quad j = 0 : m-2. \tag{52}$$

It turns out that

$$h_{k+p} = \sum_{j=1}^{mn} \sum_{i=0}^{j-1} b_j d_{k+j-i-1} h_{mn-j+p}, \quad p = 0, 1, 2, \cdots, \tag{53}$$

where $d_k = d^{(k)}(0)$ satisfies the characteristic difference equation

$$\sum_{k=0}^{mn} b_j d_{k+mn-j} = 0, \quad b_0 d_{mn-1} = 1, \quad d_k = 0, k = 0 : mn-2. \tag{54}$$

The variation of constants formula for the continous and discrete equation in terms of the matrix impulse response can be written in a straightforward manner.

7 Bibliography

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